

Bismut Formulae and Applications for Functional SPDEs*

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Abstract

By using Malliavin calculus, explicit derivative formulae are established for a class of semi-linear functional stochastic partial differential equations with additive or multiplicative noise. As applications, gradient estimates and Harnack inequalities are derived for the semigroup of the associated segment process.

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1 Introduction

The Bismut-type formulae, initiated in [4], are powerful tools to derive regularity estimates for the underlying Markov semigroups. The formulae have been developed and applied in various settings, e.g., in [6] for stochastic partial differential equations (SPDEs) driven by cylindrical Wiener processes and [7] for semi-linear SPDEs with Lévy noise, using a simple martingale approach proposed by Elworthy-Li [8]; in [15] for linear stochastic differential equations (SDEs) driven by (purely jump) Lévy processes in terms of lower bound conditions of Lévy measures; in [3, 10] for degenerate SDEs with additive noise, using a coupling technique; in [9, 11, 18, 19] for degenerate SDEs using Malliavin calculus.

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However, there are few analogues for functional SPDEs (even for finite-dimensional functional SDEs) with multiplicative noise. In this paper we aim to establish explicit Bismut-type formulae for a class of functional SPDEs with additive or multiplicative noise. Noting that for functional SDEs the martingale method used in [8] does not work due to the lack of backward Kolmogorov equation for the segment process, and the coupling method developed in [1, 3, 10, 16] seems not easy to apply provided the noise is multiplicative, we will mainly make use of Malliavin calculus.

Let $(H, \langle \cdot, \cdot \rangle, \|\cdot\|)$ be a real separable Hilbert space, and $(W(t))_{t \geq 0}$ a cylindrical Wiener process on H with respect to a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Let $\mathcal{L}(H)$ and $\mathcal{L}_{HS}(H)$ be the spaces of all linear bounded operators and Hilbert-Schmidt operators on H respectively. Denote by $\|\cdot\|$ and $\|\cdot\|_{HS}$ the operator norm and the Hilbert-Schmidt norm respectively. Let $\tau > 0$ be fixed and let $\mathcal{C} := C([- \tau, 0] \rightarrow H)$, the space of all H -valued continuous functions defined on $[- \tau, 0]$, equipped with the uniform norm $\|f\|_\infty := \sup_{-\tau \leq \theta \leq 0} \|f(\theta)\|$. For a map $h : [- \tau, \infty) \rightarrow H$ and $t \geq 0$, let $h_t \in \mathcal{C}$ be the segment of $h(t)$, i.e. $h_t(\theta) = h(t + \theta)$, $\theta \in [- \tau, 0]$.

Consider the following semi-linear functional SPDE

$$(1.1) \quad \begin{cases} dX(t) = \{AX(t) + F(X_t)\}dt + \sigma(X(t))dW(t), \\ X_0 = \xi \in \mathcal{C}, \end{cases}$$

where

- (A1) $(A, \mathcal{D}(A))$ is a linear operator on H generating a contractive C_0 -semigroup $(e^{tA})_{t \geq 0}$.
- (A2) $F : \mathcal{C} \rightarrow H$ is Fréchet differentiable such that $\nabla F : \mathcal{C} \times \mathcal{C} \rightarrow H$ is bounded on $\mathcal{C} \times \mathcal{C}$ and uniformly continuous on bounded sets.
- (A3) $\sigma : H \rightarrow \mathcal{L}(H)$ is Fréchet differentiable such that $\nabla \sigma : H \times H \rightarrow \mathcal{L}_{HS}(H)$ is bounded on $H \times H$ and uniformly continuous on bounded sets, and $\sigma(x)$ is invertible for each $x \in H$.
- (A4) $\int_0^t s^{-2\alpha} \|e^{sA} \sigma(0)\|_{HS}^2 ds < \infty$ holds for some constant $\alpha \in (0, \frac{1}{2})$ and all $t > 0$.

Recall that a mild solution is a continuous adapt process $(X(t))_{t \geq -\tau}$ on H such that

$$X(t) = e^{tA} \xi(0) + \int_0^t e^{(t-s)A} F(X_s) ds + \int_0^t e^{(t-s)A} \sigma(X(s)) dW(s), \quad t \geq 0.$$

By (A1) – (A4), equation (1.1) has a unique mild solution (see Theorem A.1 in the Appendix section), denoted by $(X^\xi(t))_{t \geq 0}$, the solution with $X_0 = \xi \in \mathcal{C}$. Let

$$P_t f(\xi) := \mathbb{E} f(X_t^\xi), \quad t \geq 0, \xi \in \mathcal{C}, f \in \mathcal{B}_b(\mathcal{C}),$$

where $\mathcal{B}_b(\mathcal{C})$ is the class of all bounded measurable functions on \mathcal{C} . We remark that due to the time-delay the solution $(X^\xi(t))_{t \geq 0}$ is not Markovian, but its segment process $(X_t^\xi)_{t \geq 0}$ admits strong Markov property, so that P_t is a Markov semigroup on $\mathcal{B}_b(\mathcal{C})$.

The following two theorems are the main results of the paper, which provide derivative formulae for P_t with additive and multiplicative noise respectively.

Theorem 1.1 (Additive Noise). *Assume that (A1)-(A4) hold with constant $\sigma \in \mathcal{L}(H)$. Then for any $T > \tau$ and $u \in C^1([0, \infty))$ such that $u(0) = 1$ and $u(t) = 0$ for $t \geq T - \tau$,*

$$(1.2) \quad \nabla_\eta P_T f(\xi) = \mathbb{E} \left(f(X_T^\xi) \int_0^T \langle \sigma^{-1}(\nabla_{\Upsilon_t} F(X_t^\xi) - \dot{u}(t)e^{tA}\eta(0)), dW(t) \rangle \right)$$

holds for all $\xi, \eta \in \mathcal{C}$ and $f \in C_b^1(\mathcal{C})$, where

$$\Upsilon(t) := \begin{cases} u(t)e^{tA}\eta(0), & t > 0, \\ \eta(t), & t \in [-\tau, 0]. \end{cases}$$

Theorem 1.2 (Multiplicative Noise). *Assume that (A1)-(A4) hold. Let $T > \tau$ and $u \in C^1([0, \infty))$ be such that $u(t) > 0$ for $t \in [0, T - \tau]$, $u(t) = 0$ for $t \geq T - \tau$, and*

$$\theta_p := \inf_{t \in [0, T-\tau]} \{p + (p-1)u'(t)\} > 0$$

holds for some $p > 1$. Then for any $\xi, \eta \in \mathcal{C}$:

(1) *The equation*

$$(1.3) \quad \begin{cases} dZ(t) = \left\{ AZ(t) + (\nabla_{Z_t} F(X_t^\xi) - \frac{Z(t)}{u(t)} 1_{[0, T-\tau)}(t) \right\} dt \\ \quad + (\nabla_{Z(t)} \sigma(X^\xi(t))) dW(t), \\ Z_0 = \eta, \end{cases}$$

has a unique solution such that $Z(t) = 0$ for $t \geq T - \tau$.

(2) *If $\|\sigma^{-1}(\cdot)\| \leq c(1 + \|\cdot\|^q)$ holds for some constants $c, q > 0$, then*

$$(1.4) \quad \begin{aligned} \nabla_\eta P_T f(\xi) = \mathbb{E} \left(f(X_T^\xi) \int_0^T \left\langle \sigma^{-1}(X^\xi(t)) \left\{ \frac{Z(t)}{u(t)} 1_{[0, T-\tau)}(t) \right. \right. \right. \\ \left. \left. \left. + \nabla_{Z_t} F(X_t^\xi) 1_{[T-\tau, T]}(t) \right\}, dW(t) \right\rangle \right) \end{aligned}$$

holds for $f \in C_b^1(\mathcal{C})$.

A simple choice of u for Theorem 1.1 is $u(t) = \frac{(T-\tau-t)^+}{T-\tau}$, while for Theorem 1.2 one may take $u(t) = (T - \tau - t)^+$ such that $\theta_p = 1$ for all $p > 1$. Both theorems will be proved in the next section. In Section 3 these results are applied to derive explicit gradient estimates and Harnack inequalities of P_t . Finally, for completeness, in the Appendix section we address the existence and uniqueness of mild solution to equation (1.1) under (A1)-(A4), and the existence of Malliavin derivative $D_h X^\xi(t)$ along direction h and derivative process $\nabla_\eta X^\xi(t)$ along direction η as solutions of SPDEs on H .

2 Proofs of Theorems 1.1 and 1.2

For the readers' convenience, let us first explain the main idea of establishing Bismut formula using Malliavin calculus. Let H_a^1 be the class of all adapted process $h = (h(t))_{t \geq 0}$ on H such that $h(0) = 0$,

$$\dot{h}(t) := \frac{d}{dt}h(t)$$

exists $\mathbb{P} \times dt$ -a.e. and

$$\mathbb{E} \int_0^T \|\dot{h}(t)\|^2 dt < \infty, \quad T > 0.$$

For $\epsilon > 0$ and $h \in H_a^1$, let $X^{\xi, \epsilon h}(t)$ solve (1.1) with $W(t)$ replaced by $W(t) + \epsilon h(t)$, i.e.,

$$(2.1) \quad \begin{cases} dX^{\xi, \epsilon h}(t) = \{AX^{\xi, \epsilon h}(t) + F(X_t^{\xi, \epsilon h}) + \epsilon \sigma(X^{\xi, \epsilon h}(t))\dot{h}(t)\}dt \\ \quad + \sigma(X^{\xi, \epsilon h}(t))dW(t), \\ X_0^{\xi, \epsilon h} = \xi \in \mathcal{C}. \end{cases}$$

If for $h \in H_a^1$

$$D_h X_t^\xi := \left. \frac{d}{d\epsilon} X_t^{\xi, \epsilon h} \right|_{\epsilon=0}$$

exists in $L^2(\Omega \rightarrow H; \mathbb{P})$, we call it the Malliavin derivative of X_t^ξ along direction h . Next, let

$$\nabla_\eta X_t^\xi := \left. \frac{d}{d\epsilon} X_t^{\xi + \epsilon \eta} \right|_{\epsilon=0}$$

be the derivative process of X_t^ξ along direction $\eta \in \mathcal{C}$. If

$$(2.2) \quad D_h X_T^\xi = \nabla_\eta X_T^\xi, \quad \text{a.s.},$$

then for any $f \in C_b^1(\mathcal{C})$

$$\begin{aligned} \nabla_\eta P_T f(\xi) &= \mathbb{E} \nabla_\eta f(X_T^\xi) = \mathbb{E} \nabla_{\nabla_\eta X_T^\xi} f(X_T^\xi) \\ &= \mathbb{E} \nabla_{D_h X_T^\xi} f(X_T^\xi) = \mathbb{E} D_h f(X_T^\xi). \end{aligned}$$

Combining this with the integration by parts formula for D_h , we obtain

$$\nabla_\eta P_T f(\xi) = \mathbb{E} \left(f(X_T^\xi) \int_0^T \langle \dot{h}(t), dW(t) \rangle \right).$$

In conclusion, the key point of the proof is, for given $T > \tau$, $\xi, \eta \in \mathcal{C}$ and $f \in C_b^1(\mathcal{C})$, to construct an $h \in H_a^1$ such that (2.2) holds.

We are now in a position to complete the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Let $h(0) = 0$ and

$$\dot{h}(t) = \sigma^{-1} \{ \nabla_{\Upsilon_t} F(X_t^\xi) - \dot{u}(t) e^{tA} \eta(0) \}, \quad t \geq 0.$$

By (A1) and $u \in C^1([0, T - \tau])$, we see that $h \in H_a^1$. Moreover, $\Upsilon(t)$ solves the equation

$$(2.3) \quad \begin{cases} d\Upsilon(t) = \{ A\Upsilon(t) + \nabla_{\Upsilon_t} F(X_t^\xi) - \sigma \dot{h}(t) \} dt, & t \geq 0, \\ \Upsilon_0 = \eta. \end{cases}$$

On the other hand, by Theorem A.2 in Appendix, when $\nabla \sigma = 0$, $\nabla_\eta X^\xi(t) - D_h X^\xi(t)$ also solves this equation. Since it is trivial that (2.3) has a unique solution, we conclude that

$$\nabla_\eta X^\xi(t) - D_h X^\xi(t) = \Upsilon(t), \quad t \geq 0.$$

Thus, $\nabla_\eta X_T^\xi = D_h X_T^\xi$ as $\Upsilon_T = 0$ according to the choice of u . Therefore, the desired derivative formula holds as explained above. \square

To prove Theorem 1.2, we need the following lemma. Since $\nabla F(X_t^\xi) : \mathcal{C} \rightarrow H$ and $\nabla \sigma(X^\xi(t)) : H \rightarrow \mathcal{L}_{HS}(H)$ are linear and bounded, (1.3) has a unique strong (variational) solution for $t \in [0, T - \tau]$.

Lemma 2.1. *In the situation of Theorem 1.2, let $(Z(t))_{t \in [0, T - \tau]}$ solve (1.3). Then for any $p > 0$ there exists a constant $C > 0$ such that*

$$\sup_{t \in [0, T - \tau]} \|Z_t\|_\infty^p < C \|\eta\|_\infty^p, \quad \eta \in \mathcal{C}.$$

Proof. It suffices to prove for $p > 2$. By Itô's formula and the boundedness of ∇F and $\nabla \sigma$, there exists a constant $c_1 > 0$ such that

$$\begin{aligned} d\|Z(t)\|^2 &= \left\{ 2 \left\langle Z(t), AZ(t) + \nabla_{Z_t} F(X_t^\xi) - \frac{Z(t)}{u(t)} \right\rangle + \|\nabla_{Z(t)} \sigma(X^\xi(t))\|_{HS}^2 \right\} dt \\ &\quad + 2 \langle Z(t), (\nabla_{Z(t)} \sigma(X^\xi(t)) dW(t)) \rangle \\ &\leq \left\{ c_1 \|Z_t\|_\infty^2 - \frac{2\|Z(t)\|^2}{u(t)} 1_{[0, T - \tau]}(t) \right\} dt + 2 \langle Z(t), (\nabla_{Z(t)} \sigma(X^\xi(t)) dW(t)) \rangle \end{aligned}$$

holds for $t \in [0, T - \tau]$. So, for $p > 2$ there exists a constant $c_2 > 0$ such that

$$(2.4) \quad \begin{aligned} d\|Z(t)\|^p &= d(\|Z(t)\|^2)^{\frac{p}{2}} \\ &= \left\{ \frac{p}{2} \|Z(t)\|^{p-2} d\|Z(t)\|^2 + \frac{p}{2} (p-2) \|Z(t)\|^{p-4} \|(\nabla_{Z(t)} \sigma(X^\xi(t)))^* Z(t)\|^2 \right\} dt \\ &\quad + p \|Z(t)\|^{p-2} \langle Z(t), (\nabla_{Z(t)} \sigma(X^\xi(t)) dW(t)) \rangle \\ &\leq \left\{ c_2 \|Z_t\|_\infty^p - \frac{p\|Z(t)\|^p}{u(t)} \right\} dt + p \|Z(t)\|^{p-2} \langle Z(t), (\nabla_{Z(t)} \sigma(X^\xi(t)) dW(t)) \rangle \end{aligned}$$

holds for $t \in [0, T - \tau]$. Since $\|\nabla_{Z(t)}\sigma(X^\xi(t))\|_{HS} \leq c\|Z(t)\|$ holds for some constant $c > 0$, combining this with the Burkhold-Davis-Gundy inequality, we arrive at

$$\mathbb{E} \sup_{s \in [-\tau, t]} \|Z(s)\|^p \leq \|\eta\|_\infty^p + c_3 \int_0^t \mathbb{E} \sup_{s \in [-\tau, \theta]} \|Z(s)\|^p ds, \quad t \in [0, T - \tau]$$

for some constant $c_3 > 0$. The proof is then completed by the Gronwall lemma. \square

Proof of Theorem 1.2. (1) Due to (A1) – (A4), it is easy to see that (1.3) has a unique solution for $t \in [0, T - \tau]$. Let

$$\tilde{Z}(t) = Z(t)1_{[-\tau, T-\tau]}(t), \quad t \geq -\tau.$$

If

$$(2.5) \quad \lim_{t \uparrow T-\tau} Z(t) = 0,$$

then it is easy to see that $(\tilde{Z}(t))_{t \geq 0}$ solves (1.3) and hence, the proof is finished. By Itô's formula and (2.4) we can deduce that

$$\begin{aligned} d \frac{\|Z(t)\|^p}{u^{p-1}(t)} &= \frac{1}{u^{p-1}(t)} d\|Z(t)\|^p - (p-1) \frac{\dot{u}(t)\|Z(t)\|^p}{u^p(t)} dt \\ &\leq -\theta_p \frac{\|Z(t)\|^p}{u^p(t)} dt + C_1 \|Z(t)\|_\infty^p dt \\ &\quad + \frac{p}{u^{p-1}(t)} \|Z(t)\|^{p-2} \langle Z(t), (\nabla_{Z(t)}\sigma(X^\xi(t))) dW(t) \rangle \end{aligned}$$

for some constant $C_1 > 0$. Combining this with Lemma 2.1 we obtain

$$(2.6) \quad \mathbb{E} \int_0^{T-\tau} \frac{\|Z(t)\|^p}{u^p(t)} dt \leq C_2 \left(\|\eta\|_\infty^p + \frac{\|\eta(0)\|^p}{u^{p-1}(0)} \right)$$

for some constant $C_2 > 0$, and due to the Burkhold-Davis-Gundy inequality

$$\mathbb{E} \sup_{s \in [0, T-\tau]} \frac{\|Z(s)\|^p}{u^{p-1}(s)} < \infty.$$

Since $u(s) \downarrow 0$ as $s \uparrow T - \tau$, the latter implies (2.5).

(2) Let

$$h(t) = \int_0^t \sigma^{-1}(X^\xi(s)) \left\{ \frac{Z(s)}{u(s)} 1_{[0, T-\tau]}(s) + \nabla_{Z_s} F(X_s^\xi) 1_{[T-\tau, T]}(s) \right\} ds, \quad t \geq 0.$$

We first prove that $h \in H_a^1$. According to the boundedness of $\|\nabla F\|$ and using the Hölder inequality, we obtain

$$\begin{aligned} \mathbb{E} \int_0^T \|\dot{h}(t)\|^2 dt &\leq \mathbb{E} \int_0^{T-\tau} \|\sigma^{-1}(X^\xi(t))\|^2 \frac{\|Z(t)\|^2}{u^2(t)} dt + C \mathbb{E} \int_{T-\tau}^T \|\sigma^{-1}(X^\xi(t))\|^2 \|Z_s\|_\infty^2 dt \\ &\leq \left(\mathbb{E} \int_0^T \|\sigma^{-1}(X^\xi(t))\|^{\frac{2p}{p-2}} dt \right)^{\frac{p-2}{p}} \times \left\{ \left(\mathbb{E} \int_0^{T-\tau} \frac{\|Z(t)\|^p}{u^p(t)} dt \right)^{\frac{2}{p}} + C \left(\mathbb{E} \int_{T-\tau}^T \|Z_t\|_\infty^p dt \right)^{\frac{2}{p}} \right\} \end{aligned}$$

for some constant $C > 0$. Combining this with (2.6), $\|\sigma^{-1}(x)\| \leq c(1 + \|x\|^q)$, Lemma 2.1 and Theorem A.1 below, we conclude that $\mathbb{E} \int_0^T \|\dot{h}(t)\|^2 dt < \infty$; that is, $h \in H_a^1$.

Next, we intend to show that $\nabla_h X_T^\xi = D_h X_T^\xi$, which implies the desired derivative formula as explained in the beginning of this section. It is easy to see from Theorem A.2 below and the definition of h that $\Gamma(t) := \nabla_\eta X^\xi(t) - D_h X^\xi(t)$ solves the equation

$$\begin{cases} d\Gamma(t) = \left\{ A\Gamma(t) + \nabla_{\Gamma_t} F(X_t^\xi) - \frac{Z(t)}{u(t)} 1_{[0, T-\tau)}(t) - \nabla_{Z_t} F(X_t^\xi) 1_{[T-\tau, T]}(t) \right\} dt \\ \quad + \nabla_{\Gamma(t)} \sigma(X^\xi(t)) dW(t), \quad t \in [0, T] \\ \Gamma_0 = \eta. \end{cases}$$

Then for $t \in [0, T]$,

$$\begin{cases} d(\Gamma(t) - Z(t)) = \left\{ A(\Gamma(t) - Z(t)) + \nabla_{\Gamma_t - Z_t} F(X_t^\xi) \right\} dt + \nabla_{\Gamma(t) - Z(t)} \sigma(X^\xi(t)) dW(t), \\ \Gamma_0 - Z_0 = 0. \end{cases}$$

By Itô's formula and using (A1)-(A3), we obtain

$$d\|\Gamma(t) - Z(t)\|^2 \leq C\|\Gamma_t - Z_t\|_\infty^2 dt + 2\langle \Gamma(t) - Z(t), \nabla_{\Gamma(t) - Z(t)} \sigma(X^\xi(t)) dW(t) \rangle$$

for some constant $C > 0$ and all $t \in [0, T]$. By the boundedness of $\|\nabla \sigma\|_{HS}$ and applying the Burkhold-Davis-Gundy inequality, we obtain

$$\mathbb{E} \sup_{s \in [0, t]} \|\Gamma(s) - Z(s)\|^2 \leq C' \int_0^t \mathbb{E} \sup_{s \in [0, r]} \|\Gamma(s) - Z(s)\|^2 dr, \quad t \in [0, T]$$

for some constant $C' > 0$. Therefore $\Gamma(t) = Z(t)$ for all $t \in [0, T]$. In particular, $\Gamma_T = Z_T$. Since $Z_T = 0$, we obtain $\nabla_\eta X_T^\xi = D_h X_T^\xi$. \square

Remark 2.1. Our main results, Theorem 1.1 and Theorem 1.2, are established under the assumption that the infinitesimal generator A generates a contractive C_0 -semigroup. Replacing A and $F(x)$ by $A - \alpha$ and $F(x) + \alpha x$ for a positive constant $\alpha > 0$, they also work for A generating a pseudo-contractive C_0 -semigroup, i.e., $\|e^{tA}\| \leq e^{\alpha t}$.

3 Gradient Estimate and Harnack Inequality

In this section we give some applications of Bismut formulae for P_t with additive and multiplicative noise respectively.

Theorem 3.1 (Additive Noise). *Assume that (A1) – (A4) hold with constant $\sigma \in \mathcal{L}(H)$. Then there exists a constant $C > 0$ such that*

(1) *For any $T > \tau, \xi, \eta \in \mathcal{C}$ and $f \in \mathcal{B}_b(\mathcal{C})$,*

$$|\nabla_\eta P_T f(\xi)|^2 \leq \frac{C}{(T - \tau) \wedge 1} P_T f^2(\xi).$$

(2) For any $T > \tau, \xi, \eta \in \mathcal{C}$ and positive $f \in \mathcal{B}_b(\mathcal{C})$,

$$(3.1) \quad |\nabla_\eta P_T f(\xi)| \leq \delta \{P_T(f \log f) - (P_T f) \log P_T f\}(\xi) + \frac{\|\eta\|_\infty^2}{\delta \{(T - \tau) \wedge 1\}} P_T f(\xi), \quad \delta > 0.$$

Proof. By the Jensen inequality and the semigroup property of P_t , it suffices to prove for $T - \tau \in (0, 1]$. Let $u(t) = \frac{(T - \tau - t)^+}{T - \tau}$. By Theorem 1.1, the proof is then standard and similar to that of [10, Theorem 4.2]. We include it below for completeness.

(1) Note that $\dot{u}(t) = -\frac{1}{T - \tau}$. Due to the definition of $\Upsilon(t)$ and the boundedness of $\|\nabla F\|$ it follows that

$$|\nabla_{\Upsilon_t} F(X_t^\xi)|^2 \leq C \|\eta\|_\infty^2$$

for some constant $C > 0$. By (1.2), Hölder's inequality and the boundedness of $\|\sigma^{-1}\|$ we have

$$(3.2) \quad \begin{aligned} |\nabla_\eta P_T f(\xi)|^2 &\leq 2P_T f^2(\xi) \mathbb{E} \int_0^T \left\{ \|\sigma^{-1} \nabla_{\Upsilon_t} F(X_t^\xi)\|^2 + \frac{1_{[0, T - \tau)}(s)}{(T - \tau)^2} \|e^{sA} \eta(0)\|^2 \right\} dt \\ &\leq \frac{C}{T - \tau} \|\eta\|_\infty^2 P_T f^2(\xi) \end{aligned}$$

for some constant $C > 0$ and all $T \in (\tau, \tau + 1]$.

(2) For $t \in [0, T]$, let

$$M(t) := \int_0^t \left\langle \sigma^{-1} \left(\nabla_{\Upsilon_s} F(X_s^\xi) + \frac{1_{[0, T - \tau)}(s)}{T - \tau} e^{sA} \eta(0) \right), dW(s) \right\rangle,$$

which is a mean-square integrable martingale, with quadratic variation process

$$\langle M \rangle(t) := \int_0^t \left\| \sigma^{-1} \left(\nabla_{\Upsilon_s} F(X_s^\xi) + \frac{1_{[0, T - \tau)}(s)}{T - \tau} e^{sA} \eta(0) \right) \right\|^2 ds \leq \frac{C \|\eta\|_\infty^2}{T - \tau}, \quad t \in [0, T]$$

for some constant $C > 0$. In the light of (1.2) and Young's inequality [2, Lemma 2.4], we have that for any $\delta > 0$ and positive $f \in \mathcal{B}_b(\mathcal{C})$

$$|\nabla_\eta P_T f(\xi)| \leq \delta \{P_T(f \log f) - (P_T f) \log P_T f\}(\xi) + \delta P_T f(\xi) \log \mathbb{E} \exp \left(\frac{1}{\delta} M(T) \right).$$

Moreover, by the exponential martingale inequality, the boundedness of $\|\nabla F\|$ and the definition of Υ_s ,

$$\mathbb{E} \exp \left(\frac{1}{\delta} M(T) \right) \leq \left\{ \mathbb{E} \exp \left(\frac{2}{\delta^2} \langle M \rangle(T) \right) \right\}^{\frac{1}{2}} \leq \exp \left(\frac{C}{\delta^2 (T - \tau)} \|\eta\|_\infty^2 \right)$$

holds for some constant $C > 0$ and all $T \in (\tau, \tau + 1]$. Therefore, the proof is finished. \square

According to [10, Proposition 4.1], (3.1) implies the following Harnack inequality. Applications of these inequalities to heat kernel estimates, invariant probability measure and Entropy-cost inequalities can be found in e.g. [12, 13, 15].

Corollary 3.2. *Assume that (A1) – (A4) hold with constant $\sigma \in \mathcal{L}(H)$. Then there exists a constant $C > 0$ such that*

$$(3.3) \quad |P_T f|^\alpha(\xi) \leq \exp \left[\frac{\alpha C \|\eta\|_\infty^2}{(\alpha - 1)\{(T - \tau) \wedge 1\}} \right] P_T |f|^\alpha(\xi + \eta), \quad f \in \mathcal{B}_b(\mathcal{C}), T > \tau, \xi, \eta \in \mathcal{C}$$

holds for any $\alpha > 1$.

Next, we consider the multiplicative noise case. For simplicity we only consider the case where $\|\sigma^{-1}\|_\infty := \sup_{x \in H} \|\sigma^{-1}(x)\| < \infty$. The case for σ^{-1} having algebraic growth is similar, where the resulting estimate of $\|\nabla P_t f\|$ will be no longer bounded for bounded f , but bounded above by a polynomial function of $\|\xi\|_\infty$.

Theorem 3.3 (Multiplicative Noise). *Let Assume (A1)-(A4) and assume that $\|\sigma^{-1}\|_\infty < \infty$. Then for any $p > 1$ there exists a constant $C > 0$ such that*

$$|\nabla_\eta P_T f(\xi)| \leq \frac{C \|\eta\|_\infty}{1 \wedge \sqrt{T - \tau}} (P_T |f|^p)^{\frac{1}{p}}(\xi), \quad f \in \mathcal{B}_b(\mathcal{C}), T > \tau, \xi, \eta \in \mathcal{C}.$$

In particular, P_t is strong Feller for $t > T - \tau$.

Proof. It suffices to prove for $T \in (\tau, \tau + 1]$. Let $u(t) = (T - \tau - t)^+, t \geq 0$. We have $\theta_p = 1$. Since σ^{-1} is bounded, for any $p > 1$ and $\eta \in \mathcal{C}$, it follows from (1.4) that

$$\begin{aligned} & \frac{|\nabla_\eta P_T f|^{\frac{p}{p-1}}(\xi)}{(P_T |f|^p)^{\frac{1}{p-1}}(\xi)} \\ & \leq \mathbb{E} \left| \int_0^T \left\langle \sigma^{-1}(X^\xi(s)) \left\{ \frac{Z(s)}{u(s)} 1_{[0, T-\tau)}(s) + \nabla_{Z_s} F(X_s^\xi) 1_{[T-\tau, T]}(s) \right\}, dW(s) \right\rangle \right|^{\frac{p}{p-1}} \\ & \leq C_1 \mathbb{E} \left(\int_0^T \left(\frac{|Z(t)|^2}{u^2(t)} 1_{[0, T-\tau)}(t) + \|Z_t\|_\infty^2 1_{[T-\tau, T]}(t) \right) dt \right)^{\frac{p}{2(p-1)}} \end{aligned}$$

holds for some constants $C_1, C_2 > 0$ and all $T \in (\tau, \tau + 1]$, where the second inequality follows from the Burkholder-Davis-Gundy inequality: for any $q > 1$ there exists a constant $C_q > 0$ such that

$$\mathbb{E} \sup_{t \in [0, T]} |M(t)|^q \leq C_q \mathbb{E} \langle M \rangle^{\frac{q}{2}}(T)$$

holds for any continuous martingale $M(t)$ and $T > 0$. Then the proof is completed by combining this with (2.6) with $u(0) = T - \tau$ and Lemma 2.1. \square

Remark 3.1. From Corollary 3.2 and [10, Proposition 4.1], we know that entropy estimation (3.1) plays a key role in establishing the Harnack inequality. However, the entropy estimation seems to be difficult to obtain for the multiplicative noise case. Hence we can not adopt the same method as in the additive noise case to derive the Harnack inequality. In order to establish the Harnack inequality for the multiplicative noise case, one may use coupling method as in Wang [14], and Wang and Yuan [17]. Since the derivation of the Harnack inequality for functional SPDEs with multiplicative noise is very similar to that of [17], we omit it here.

A Appendix

In this section we give two auxiliary lemmas, where one concerns the existence and uniqueness of solution of equation (1.1) under (A1)-(A4), and the other one discusses not only the existence of Malliavin directional derivative but also the derivative process with respect to the initial data. To make the content self-contained, we sketch their proofs.

Theorem A.1. *Let (A1), (A4) hold, and let $F : H \rightarrow H, \sigma : H \rightarrow \mathcal{L}(H)$ be Lipschitz continuous. Then for any $p > 2$ and initial data $\xi \in L^p(\Omega \rightarrow \mathcal{C}, \mathcal{F}_0, \mathbb{P})$, equation (1.1) has a unique mild solution $(X^\xi(t))_{t \geq 0}$, and the solution satisfies*

$$\mathbb{E} \sup_{t \in [0, T]} \|X_t^\xi\|_\infty^p < \infty, \quad T > 0.$$

Proof. Obviously, (A4) remains true by replacing α with a smaller positive number. So, we may take in (A4) $\alpha \in (0, \frac{1}{p})$. Then, by [5, Proposition 7.9] for $r = \frac{p}{2} \in (1, \frac{1}{2\alpha})$, for any $T_0 > 0$ there exists a constant $C_0 > 0$ such that for any continuous adapted process $Y(s)$ on H ,

$$(A.1) \quad \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t e^{(t-s)A} \sigma(Y(s)) dW(s) \right|^p \leq C_0 \mathbb{E} \int_0^T \|\sigma(Y(s))\|^p ds, \quad T \in [0, T_0].$$

Using this inequality, the desired assertions follow from the classical fixed point theorem for contractions. Denote by \mathcal{H}_p the Banach space of all the H -valued continuous adapted processes Y defined on the time interval $[-\tau, T]$ such that $Y(t) = \xi(t), t \in [-\tau, 0]$, and

$$\|Y\|_p := \left(\mathbb{E} \sup_{t \in [-\tau, T]} \|Y(t)\|^p \right)^{\frac{1}{p}} < \infty.$$

Let

$$\mathcal{K}(Y)(t) = \begin{cases} \xi(t), & \text{if } t \in [-\tau, 0], \\ e^{tA} \xi(0) + \int_0^t e^{(t-s)A} F(Y_s) ds + \int_0^t e^{(t-s)A} \sigma(Y(s)) dW(s), & \text{if } t \in (0, T]. \end{cases}$$

By (A.1) and the linear growth of F and σ , we conclude that \mathcal{K} maps \mathcal{H}_p into \mathcal{H}_p . For the existence and uniqueness of solutions, it suffices to show that the map \mathcal{K} is contractive for small $T > 0$. By the Lipschitz continuity of F and σ , and applying (A.1) for $\sigma(Y_1(s)) - \sigma(Y_2(s))$ in place of $\sigma(Y(s))$, we obtain

$$\|\mathcal{K}(Y^1) - \mathcal{K}(Y^2)\|_p \leq CT \|Y^1 - Y^2\|_p^p, \quad Y^1, Y^2 \in \mathcal{H}_p.$$

for some constant $C > 0$ and all $T \in [0, T_0]$. Choosing sufficiently small T such that $CT < 1$ we can conclude that \mathcal{K} is contractive. \square

Theorem A.2. *Assume that (A1), (A2) and (A3) hold, and let $\xi, \eta \in \mathcal{C}$ and $h \in H_a^1$.*

(1) $(D_h X(t))_{t \geq 0}$ exists and is the unique solution to the equation

$$\begin{cases} d\alpha(t) = \{A\alpha(t) + \nabla_{\alpha_t} F(X_t^\xi) + \sigma(X^\xi(t))\dot{h}(t)\}dt \\ \quad + (\nabla_{\alpha(t)} \sigma(X^\xi(t)))dW(t), \\ \alpha_0 = 0. \end{cases}$$

(2) $(\nabla_\eta X(t))_{t \geq 0}$ exists and is the unique solution to the equation

$$\begin{cases} d\beta(t) = \{A\beta(t) + \nabla_{\beta_t} F(X_t^\xi)\}dt + (\nabla_{\beta(t)} \sigma(X^\xi(t)))dW(t), \\ \beta_0 = \eta. \end{cases}$$

Proof. We only prove (1) since (2) can be proved in a similar way. The argument of the proof is standard in the setting of semi-linear SPDEs without delay. The only difference for the present setting is that one has to estimate the sup over time for the norm of the error process for small $\epsilon \in (0, 1)$

$$\Lambda^\epsilon(t) := X^{\xi, \epsilon h}(t) - X^\xi(t) - \epsilon \alpha(t), \quad t \geq 0,$$

where $X^{\xi, \epsilon h}$ is the mild solution to (2.1).

(a) There exists a constant $C > 0$ such that

$$(A.2) \quad \mathbb{E} \sup_{t \in [0, T]} \|X_t^{\xi, \epsilon h} - X_t^\xi\|_\infty^2 \leq \epsilon^2 e^{C(T+1)} \mathbb{E} \int_0^T \|\dot{h}(t)\|^2 dt, \quad T \geq 0.$$

Indeed, by (A1), (A2) and (A3) we have the following Itô's formula for $\|X^{\xi, \epsilon h}(t) - X^\xi(t)\|^2$:

$$\begin{aligned} d\|X^{\xi, \epsilon h}(t) - X^\xi(t)\|^2 &= 2\langle X^{\xi, \epsilon h}(t) - X^\xi(t), A(X^{\xi, \epsilon h}(t) - X^\xi(t)) \\ &\quad + F(X_t^{\xi, \epsilon h}) - F(X_t^\xi) + \epsilon \sigma(X^{\xi, \epsilon h}(t))\dot{h}(t) \rangle dt \\ &\quad + \|\sigma(X^{\xi, \epsilon h}(t)) - \sigma(X^\xi(t))\|_{HS}^2 dt \\ &\quad + 2\langle X^{\xi, \epsilon h}(t) - X^\xi(t), (\sigma(X^{\xi, \epsilon h}(t)) - \sigma(X^\xi(t)))dW(t) \rangle. \end{aligned}$$

Noting from (A1), (A2) and (A3) that

$$\begin{aligned} \langle X^{\xi, \epsilon h}(t) - X^\xi(t), A(X^{\xi, \epsilon h}(t) - X^\xi(t)) \rangle &\leq 0, \\ \|F(X_t^{\xi, \epsilon h}) - F(X_t^\xi) + \epsilon \sigma(X^{\xi, \epsilon h}(t))\dot{h}(t)\| &\leq C_1(\|X_t^{\xi, \epsilon h} - X_t^\xi\|_\infty + \epsilon \|\dot{h}(t)\|), \end{aligned}$$

and by the Burkhold-Davis-Gundy inequality

$$\begin{aligned} \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t \langle X^{\xi, \epsilon h}(s) - X^\xi(s), (\sigma(X^{\xi, \epsilon h}(s)) - \sigma(X^\xi(s)))dW(s) \rangle \right| \\ \leq C_1 \mathbb{E} \left(\int_0^T \|X^{\xi, \epsilon h}(s) - X^\xi(s)\|^4 ds \right)^{\frac{1}{2}} \\ \leq \frac{1}{2} \mathbb{E} \sup_{t \in [0, T]} \|X^{\xi, \epsilon h}(t) - X^\xi(t)\|^2 + \frac{C_1}{2} \mathbb{E} \int_0^T \|X^{\xi, \epsilon h}(s) - X^\xi(s)\|^2 ds \end{aligned}$$

for some constant $C_1 > 0$, we obtain

$$\mathbb{E} \sup_{t \in [0, T]} \|X_t^{\xi, \epsilon h} - X_t^\xi\|_\infty^2 \leq C_2 \epsilon^2 \int_0^T \|\dot{h}(t)\|^2 dt + C_2 \int_0^T \mathbb{E} \sup_{s \in [0, t]} \|X_s^{\xi, \epsilon h} - X_s^\xi\|_\infty^2 dt$$

for some constant $C_2 > 0$. This implies (A.2).

(b) To prove $D_h X^\xi(t) = \alpha(t)$ it suffices to show

$$(A.3) \quad \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{E} \sup_{t \in [0, T]} \|\Lambda^\epsilon(t \wedge \tau_n)\| = 0, \quad n \geq 1,$$

where $\tau_n := \inf\{t \geq 0, \|X_t^\xi\|_\infty \geq n\} \uparrow \infty$ as $n \uparrow \infty$. To this end, we observe that

$$(A.4) \quad \begin{aligned} \Lambda^\epsilon(t \wedge \tau_n) &= \int_0^{t \wedge \tau_n} e^{(t-s)A} \{F(X_s^{\xi, \epsilon h}) - F(X_s^\xi) - \epsilon \nabla_{\alpha_s} F(X_s^\xi) \\ &\quad + \epsilon(\sigma(X_s^{\xi, \epsilon h}(s)) - \sigma(X_s^\xi(s)))\dot{h}(s)\} ds \\ &\quad + \int_0^{t \wedge \tau_n} e^{(t-s)A} (\sigma(X_s^{\xi, \epsilon h}(s)) - \sigma(X_s^\xi(s)) - \epsilon \nabla_{\alpha(s)} \sigma(X_s^\xi(s))) dW(s). \end{aligned}$$

Let

$$\gamma_n(s) := \sup_{\|\xi\|_\infty \leq n, \|\xi - \eta\|_\infty \leq s} \|\nabla F(\xi) - \nabla F(\eta)\|_\infty + \sup_{\|x\| \leq n, \|x - y\| \leq s} \|\nabla \sigma(x) - \nabla \sigma(y)\|_{HS}.$$

By (A2) and (A3) we have $\gamma_n(s) \downarrow 0$ as $s \downarrow 0$ and $\gamma_n(\infty) < \infty$. Then

$$s\gamma_n(s) \leq \gamma_n(\sqrt{\epsilon})s + \frac{s^2\gamma_n(\infty)}{\sqrt{\epsilon}}, \quad s \geq 0.$$

Therefore, there exists a constant $C_1 > 0$ such that

$$\begin{aligned} &\|F(X_s^{\xi, \epsilon h}) - F(X_s^\xi) - \epsilon \nabla_{\alpha_s} F(X_s^\xi)\|_\infty \\ &\leq \|\nabla F\|_\infty \|\Lambda_s^\epsilon\|_\infty + \|X_s^{\xi, \epsilon h} - X_s^\xi\|_\infty \gamma_n(\|X_s^{\xi, \epsilon h} - X_s^\xi\|_\infty) \\ &\leq C_1 \|\Lambda_s^\epsilon\|_\infty + \gamma(\sqrt{\epsilon}) \|X_s^{\xi, \epsilon h} - X_s^\xi\|_\infty + \frac{\gamma_n(\infty)}{\sqrt{\epsilon}} \|X_s^{\xi, \epsilon h} - X_s^\xi\|_\infty^2, \\ &\epsilon \|(\sigma(X_s^{\xi, \epsilon h}(s)) - \sigma(X_s^\xi(s)))\dot{h}(s)\| \leq \epsilon^2 \|\dot{h}(s)\|^2 + C_1 \|X_s^{\xi, \epsilon h}(s) - X_s^\xi(s)\|^2, \end{aligned}$$

and by the Burkhold-Davis-Gundy inequality

$$\begin{aligned}
& \mathbb{E} \sup_{t \in [0, T]} \left\| \int_0^{t \wedge \tau_n} e^{(t-s)A} (\sigma(X^{\xi, \epsilon h}(s)) - \sigma(X^\xi(s)) - \epsilon \nabla_{\alpha(s)} \sigma(X^\xi(s))) dW(s) \right\| \\
& \leq 2\mathbb{E} \left(\int_0^{T \wedge \tau_n} \|\sigma(X^{\xi, \epsilon h}(s)) - \sigma(X^\xi(s)) - \epsilon \nabla_{\alpha(s)} \sigma(X^\xi(s))\|_{HS}^2 ds \right)^{\frac{1}{2}} \\
& \leq 2\mathbb{E} \left(\int_0^{T \wedge \tau_n} (\|\nabla \sigma\| \|\Lambda^\epsilon(s)\| + \|X^{\xi, \epsilon h}(s) - X^\xi(s)\| \gamma_n(\|X^{\xi, \epsilon h}(s) - X^\xi(s)\|))^2 ds \right)^{\frac{1}{2}} \\
& \leq \frac{1}{2} \mathbb{E} \sup_{t \in [0, T]} \|\Lambda^\epsilon(t \wedge \tau_n)\| + \gamma_n(\sqrt{\epsilon}) \mathbb{E} \sup_{t \in [0, T]} \|X^{\xi, \epsilon h}(t \wedge \tau_n) - X^\xi(t \wedge \tau_n)\| \\
& \quad + \frac{\gamma_n(\infty)}{\sqrt{\epsilon}} \mathbb{E} \sup_{t \in [0, T]} \|X^{\xi, \epsilon h}(t \wedge \tau_n) - X^\xi(t \wedge \tau_n)\|^2 \\
& \quad + C_1 \mathbb{E} \int_0^{T \wedge \tau_n} \left(\|\Lambda^\epsilon(s)\| + \gamma_n(\sqrt{\epsilon}) \|X^{\xi, \epsilon h}(s) - X^\xi(s)\| \right. \\
& \quad \left. + \frac{\gamma_n(\infty)}{\sqrt{\epsilon}} \|X^{\xi, \epsilon h}(s) - X^\xi(s)\|^2 \right) ds.
\end{aligned}$$

Combining this with (A.2) and (A.4) we obtain

$$\mathbb{E} \sup_{t \in [0, T]} \|\Lambda^\epsilon(t \wedge \tau_n)\| \leq C_2 \int_0^T \mathbb{E} \sup_{s \in [0, t]} \|\Lambda^\epsilon(s \wedge \tau_n)\| ds + C(T) \left(\gamma_n(\sqrt{\epsilon}) \epsilon + \frac{\gamma_n(\infty) \epsilon^2}{\sqrt{\epsilon}} \right)$$

for some constant $C_2 > 0$ and

$$C(T) := e^{C_2(1+T)} \left(1 + \mathbb{E} \int_0^T \|\dot{h}(t)\|^2 dt \right), \quad T \geq 0.$$

Due to the Gronwall inequality, this implies (A.3). □

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